

THE PERPENDICULAR BISECTOR CONSTRUCTION IN n -DIMENSIONAL EUCLIDEAN AND NON-EUCLIDEAN GEOMETRIES

EMMANUEL TSUKERMAN

ABSTRACT. The “Perpendicular Bisectors Construction” is a natural way to seek a replacement for the circumcenter of a noncyclic quadrilateral in the plane. In this paper, we generalize this iterative construction to a construction on polytopes with n vertices in $(n-2)$ -dimensional Euclidean, Hyperbolic and Elliptic geometries. We then show that a number of nice properties concerning this iterative construction continue to hold in these geometries. We also introduce an analogue of the isoptic point of a quadrilateral, which is the limit point of the Perpendicular Bisectors Construction, in \mathbb{R}^n and prove some of its properties.

1. BACKGROUND

A natural way to seek a replacement for the circumcenter of a cyclic planar quadrilateral in the case when the quadrilateral is noncyclic is to proceed with the following iterative construction:

- For every 3 vertices of a quadrilateral $Q^{(1)}$, determine the circumcenter. The resulting 4 points form a new quadrilateral $Q^{(2)}$. The construction can then be iterated on $Q^{(2)}$ and then on $Q^{(3)}$, etc.

This construction is known as the “Perpendicular Bisectors Construction” since the sides of $Q^{(i+1)}$ are determined using the perpendicular bisectors of the sides of $Q^{(i)}$.

The construction is so natural that it was looked at before a number of times. In particular, the following problem about the Perpendicular Bisectors Construction was proposed by Josef Langr [1] in 1953:

The perpendicular bisectors of the sides of a quadrilateral $ABCD$ form a quadrilateral $A_1B_1C_1D_1$ and the perpendicular bisectors of the sides of $A_1B_1C_1D_1$ form a quadrilateral $A_2B_2C_2D_2$. Show that $A_2B_2C_2D_2$ is similar to $ABCD$ and find the ratio of similitude.

Given that the problem is relatively simple, it is surprising that no solutions were published in English for over half a century. The problem was mentioned by C.S. Ogilvy ([2], p. 80) as an example of an unsolved problem. According to an article on Alexander Bogomolny’s *Cut-the-knot* website

[3], “B. Grünbaum [4] wrote about the problem in 1993 as an example of an unproven problem whose correctness could not be doubted... [D. Schattschneider] proved several particular cases of the problem, but the general problem remained yet unsolved. It looks like, by that time, the problem made it into the mathematical folklore. It reached Dan Bennett by the word of mouth and its simplicity had piqued his interest. He published a solution [5] in 1997 to a major part of the problem under an additional assumption that was promptly removed by J. King [6] who (independently) also supplied a proof based on the same ideas”. A paper by G.C. Shepard [7] also found an expression for the ratio, and several simpler forms of the expression are given by Radko and Tsukerman in [8].

In the same paper, Radko and Tsukerman show that the construction (or, if $ABCD$ is non-convex, the reverse construction) has a limit called the *isoptic point*, due to its property of “being seen” at equal angles from each of the triad circles of the quadrilateral. This point has many beautiful properties, such as having a parallelogram pedal, being the unique intersection of the 6 circles of similitude of a quadrilateral and having many of the properties expected of a replacement of the circumcenter.

2. MAIN RESULTS

We introduce a generalization to the Perpendicular Bisectors Construction, which we apply to polytopes with n vertices in $(n - 2)$ -dimensional Euclidean, Hyperbolic and Elliptic geometries. We prove the remarkable property that for any dimension and any geometry previously mentioned, the i th generation polyope $P^{(i)}$ and $(i + 2)$ th generation polytope $P^{(i+2)}$ are in perspective for each i . After showing how the iterative construction in any of the geometries can be reversed via isogonal conjugation, we show that in the case of Euclidean geometry, all $P^{(2k)}$ are homothetic and all $P^{(2k+1)}$ are homothetic, and the center of homothecy is the same for both families of polytopes. Finally, we define an analogue of the isoptic point in \mathbb{R}^n and prove some of its properties.

3. PRELIMINARIES AND NOTATION

We consider d -dimensional Euclidean, Hyperbolic or Elliptic space, where $d = n - 2$. Recall that a hyperplane is a $(d - 1)$ -flat and that the mediator hyperplane of a segment P_1P_2 , denoted $PB(P_1P_2)$ throughout, is the hyperplane passing through the midpoint of P_1P_2 orthogonal to that segment. By a hypersphere, we will specifically mean a $(d - 1)$ -sphere. A facet of a polytope is a face with affine dimension $d - 1$.

Our approach to proving the perspectivity $P^{(i)}$ and $P^{(i+2)}$ will naturally involve projective geometry. Specifically, we will view Euclidean, Hyperbolic and Elliptic geometries as embedded inside of real projective d -space. For the convenience of the reader, we now give a brief overview of how to do so.

Recall that a correlation in real projective n -space $\mathbb{R}P^n$ is a one-to-one linear transformation taking points into hyperplanes and vice versa. A *polarity* is an involutory correlation, and we call the image of a point P under a polarity its *polar*, and the image of a hyperplane q its *pole*. We shall utilize the following facts:

- (1) A point P is on the polar of a point Q under a given polarity if and only if Q is on the polar of P under this same polarity. Similarly, a hyperplane p is incident to the pole of hyperplane q if and only if q is incident to the pole of p . We call such P and Q *conjugate points* and such p and q *conjugate hyperplanes*. A point that lies on its own polar is called a *self-conjugate point*. Similarly, a hyperplane incident with its own pole is a *self-conjugate hyperplane*.
- (2) A nonempty set of self-conjugate points with respect to a given polarity is a quadric and any quadric is a set of self-conjugate points with respect to some polarity.

As an illustration, given a point P outside of a conic in the projective plane, there are two tangents passing through P . The polar of P is the line incident to the two points of tangency.

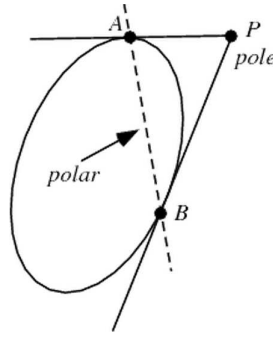


FIGURE 3.1. Pole-polar relation in the projective plane [9].

To obtain Euclidean, Hyperbolic and Elliptic geometries as subgeometries of projective geometry we fix a polarity and an associated quadric Γ , depending on the geometry. We make the following identifications in the projective plane, and the more general identification for projective n -space are similar.

- (1) For Hyperbolic geometry, the points inside of the quadric are the (ordinary) points of the geometry, points on the quadric are *ideal points* and points outside of the quadric are

hyperideal points. Hyperbolic lines are the parts of the projective lines having ordinary points.

Two hyperbolic lines are parallel (ultraparallel) if the corresponding projective lines intersect in ideal (ultraideal) points. They are perpendicular if they are conjugate with respect to Γ .

(2) For Elliptic geometry, the ordinary points are the points of the projective plane and the lines are the lines of the projective plane. Two elliptic lines are perpendicular if the corresponding projective lines are conjugate with respect to Γ .

(3) In Euclidean geometry, the ordinary points are the points of the projective plane not on Γ and the ideal points are the points on Γ . Two lines are perpendicular if their ideal points correspond under the absolute projectivity.

We refer the reader to [10] for a more comprehensive discussion on the subgeometries of projective space, e.g., on the definition of angles, distances, etc.

We list here some of the notation which will be employed throughout:

$N^{(i)}$ is the i th generation set of vertices constructed via the iterative process.

$(P_1 \cdots P_{n-1})$ will denote the unique hypersphere through points P_1, \dots, P_{n-1} .

$PB(P_1P_2)$ will denote the mediator of line segment P_1P_2 .

$PB(H_1, H_2)$ will denote a common perpendicular of hyperplanes H_1 and H_2 . In Euclidean geometry, this will simply mean that $PB(H_1, H_2)$ is perpendicular to both H_1 and H_2 .

$Iso_{P_1 \dots P_{n-1}} P_n$ denotes the isogonal conjugate of the point P_n in the simplex $P_1 \cdots P_{n-1}$.

$P^{(i)} \sim P^{(j)}$ denotes that a polytope $P^{(i)}$ with vertices $N^{(i)}$ can be chosen to have the same combinatorial type as a polytope $P^{(j)}$ with vertices $N^{(j)}$, and $P^{(i)}$ and $P^{(j)}$ are similar.

$|P^{(i)}|$ denotes the volume of $P^{(i)}$.

4. THE GENERALIZED ITERATIVE PROCESS

Consider a set $N^{(1)}$ of n points V_1, V_2, \dots, V_n in $(n-2)$ -dimensional space V . For convenience, we will say that $V_i = V_{n+i}$ for each i . When V is Euclidean geometry, we will require that any $n-1$ be affinely independent and in Hyperbolic geometry, we will also require that any $n-1$ can be circumscribed in a hypersphere, i.e. the circumcenter of the hypersphere is an ordinary, rather than ideal or hyperideal, point. Our generalization of the iterative process is as follows.

- For each vertex V_i , $i = 1, \dots, n$, construct the center $V_i^{(2)}$ of a hypersphere $(V_{i+1} \cdots V_{i+n-1})$.

The vertices $V_i^{(2)}$, $i = 1, \dots, n$, determine a new set of n points, which we will denote by $N^{(2)}$.

- The construction is then repeated on $N^{(2)}$ to produce $N^{(3)}$, etc.

It is easy to see that $N^{(2)}$ degenerates to a single point if and only if the points of $N^{(1)}$ are conhyperc-spherical, meaning that they can be inscribed in a hypersphere.

Moreover,

Lemma 1. *In $(n - 2)$ -dimensional Euclidean geometry, the set $N^{(2)}$ contains a point at infinity (an ideal point) if and only if some $n - 1$ points of $N^{(1)}$ are affinely dependent.*

Proof. Assume that some $n - 1$ points $V_i = (x_{i1}, \dots, x_{id})$, $i = 1, \dots, n - 1$, of $N^{(1)}$ are affinely dependent. The equation of the hypersphere passing through such $n - 1$ points is given by

$$\left| \begin{pmatrix} x_1^2 + x_2^2 + \dots + x_d^2 & x_1 & x_2 & \cdot & \cdot & \cdot & x_d & 1 \\ x_{11}^2 + x_{12}^2 + \dots + x_{1d}^2 & x_{11} & x_{12} & \cdot & \cdot & \cdot & x_{1d} & 1 \\ x_{21}^2 + x_{22}^2 + \dots + x_{2d}^2 & x_{21} & x_{22} & \cdot & \cdot & \cdot & x_{2d} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{n-1,1}^2 + x_{n-1,2}^2 + \dots + x_{n-1,d}^2 & x_{n-1,1} & x_{n-1,2} & \cdot & \cdot & \cdot & x_{n-1,d} & 1 \end{pmatrix} \right| = 0$$

By expanding minors across the first row, we can find the coefficient of the quadratic terms of the hypersphere to be

$$\left| \begin{pmatrix} x_{11} & x_{12} & \cdot & \cdot & \cdot & x_{1d} & 1 \\ x_{21} & x_{22} & \cdot & \cdot & \cdot & x_{2d} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{n-1,1} & x_{n-1,2} & \cdot & \cdot & \cdot & x_{n-1,d} & 1 \end{pmatrix} \right|,$$

which is zero, so the center of the hypersphere is an ideal point.

Conversely, if the center of the hypersphere is ideal, the coefficient of the quadratic terms must be zero. □

We will call $N^{(k)}$ degenerate if it contains $n - 1$ points which are affinely dependent.

Lemma 2. *In $(n - 2)$ -dimensional Euclidean geometry, if the set $N^{(1)}$ is nondegenerate and is not conhyperc-spherical, then any $n - 1$ points of $N^{(2)}$ are affinely independent.*

Proof. For convenience, we denote the hypersphere $(V_{i+1} \cdots V_{n+i-1})$ by $(V_i^{(2)})$. Assume by contradiction that some $n-1$ points $V_1^{(2)}, V_2^{(2)}, \dots, V_{n-1}^{(2)}$ of $N^{(2)}$ are affinely dependent. Then they must lie on a $(d-1)$ -flat. We can then set up our coordinate system so that hypersphere $(V_i^{(2)})$ has the expression

$$(x_1 - c_{ix_1})^2 + (x_2 - c_{ix_2})^2 + \dots + (x_{d-2} - c_{ix_{d-2}})^2 + (x_{d-1} - c_{ix_{d-1}})^2 + x_d^2 = r_i^2.$$

The intersection of any two hyperspheres $(V_i^{(2)}) \cap (V_j^{(2)})$ contains $n-2$ points from $N^{(1)}$, distinct by hypothesis, so that the hyperspheres are non-tangential. In addition, the intersection lies on a hyperplane of the form

$$x_1(2c_{ix_1} - 2c_{jx_1}) + \dots + x_{d-1}(2c_{ix_{d-1}} - 2c_{jx_{d-1}}) = r_j^2 - r_i^2 + (c_{ix_1}^2 - c_{jx_1}^2) + \dots + (c_{ix_{d-1}}^2 - c_{jx_{d-1}}^2).$$

It easy to see then that the points $V_{n-2}, V_{n-1}, V_n \in \bigcap_{i=1}^{d-1} (V_i^{(2)})$ lie on a 2-flat parallel to the 2-flat on which $V_1, V_{n-1}, V_n \in \bigcap_{i=2}^d (V_i^{(2)})$ lie. Since the two planes intersect, they must be equal. Therefore $V_1, V_{n-2}, V_{n-1}, V_n$ are affinely dependent, a contradiction. \square

From now on, we will tacitly assume that $N^{(1)}$ is nondegenerate and not conhyperspherical.

Our approach to proving the perspectivity $N^{(i)}$ and $N^{(i+2)}$ will naturally be through projective geometry. Specifically, we will view Euclidean, Hyperbolic and Elliptic geometries as embedded inside of real projective n -space. See the preliminaries section for an overview of the relevant facts on projective geometry.

Let Γ be a quadric in real projective $(n-2)$ -space $\mathbb{R}P^{n-2}$. Choose a polarity that fixes Γ . Let H_i and H'_i for $i = 1, \dots, m$ be m pairs of distinct hyperplanes and let $H_i \cap H'_i = h_i$, $i = 1, \dots, m$. In addition, let H''_i be the polar of h_i for each $i = 1, \dots, m$. We then have

Lemma 3. *The m $(n-4)$ -flats h_1, \dots, h_m lie on a hyperplane if and only if the m lines H''_1, \dots, H''_m are concurrent.*

Proof. Assume first that h_1, \dots, h_m lie on a hyperplane L . Then L is a conjugate hyperplane with respect to each H''_i . Therefore the H''_i all pass through the pole of L , which is a point.

Conversely, assume that the H''_i are concurrent at a point P . Then P is conjugate to each h_i , so the h_i all lie on the polar of P , which is a hyperplane. \square

The analogue of lemma 3 in Euclidean geometry that is of interest to us is the following trivial statement. As before, we have m pairs of hyperplanes H_i and H'_i for $i = 1, \dots, m$. Then H_i and H'_i are

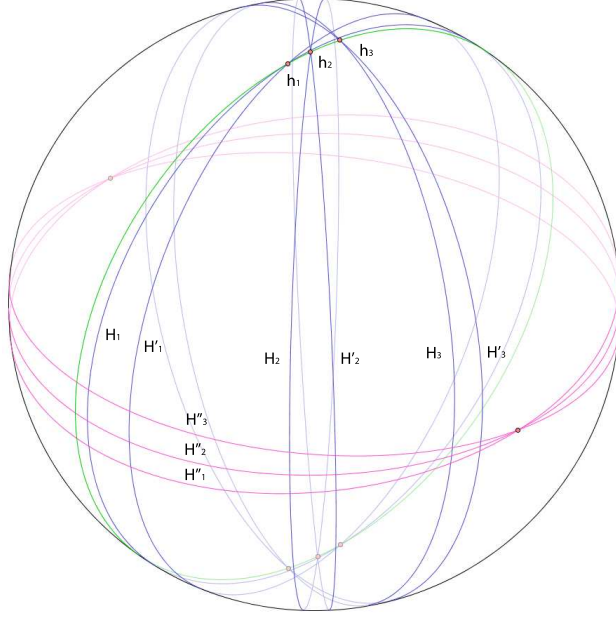


FIGURE 4.1. Lemma 3 on \mathbb{S}^2 . Each H''_i ($i = 1, 2, 3$) is the common perpendicular of H_i and H'_i . The points h_1, h_2, h_3 are collinear implying that H''_1, H''_2, H''_3 are concurrent, and conversely.

parallel if and only if some m lines H''_1, \dots, H''_m , with each H_i perpendicular to both H_i and H'_i , are concurrent. We are now ready to prove the following Theorem:

Theorem 4. *In $(n-2)$ -dimensional Euclidean, Hyperbolic and Elliptic geometries, the sets of n points $N^{(k)}$ and $N^{(k+2)}$ are perspective in a point.*

Proof. Without loss of generality, assume that $k = 1$. For simplicity, we will denote $N^{(1)}$ by N , and the points $V_i^{(1)}$ similarly. Let $N_{a,b} = \{V_1, V_2, \dots, V_n\} \setminus \{V_a, V_b\}$ and let $H_{a,b}$ be the supporting hyperplane of $N_{a,b}$. Define $H_{a,b}^{(2)}$ similarly. By construction, line $V_a^{(1)}V_b^{(1)}$ is a common perpendicular to $H_{a,b}$ and $H_{a,b}^{(2)}$. As we vary $b \in \{1, \dots, n\} \setminus \{a\}$, we obtain $n-1$ such lines all concurrent at point $V_a^{(1)}$. By the converse of lemma 3 with $m = n-1$, the elements of the set $\{H_{a,b} \cap H_{a,b}^{(2)} | b \in \{1, \dots, n\} \setminus \{a\}\}$ lie on a hyperplane. Now consider the simplices $S_a = V_{a+1} \cdots V_{n+a-1}$ and $S_a^{(2)} = V_{a+1}^{(2)} \cdots V_{n+a-1}^{(2)}$. The facets of S_a and $S_a^{(2)}$ are $H_{a,b}$ and $H_{a,b}^{(2)}$ with $b \neq a$, respectively. We apply the generalized Desargues theorem for d -dimensional space to the two simplices (see [11]), to conclude that they are perspective in a point. Call this point W . By considering another pair of simplices, we conclude that they too must be perspective in W , because the simplices of the same generation share parts, so that N and $N^{(2)}$ are in perspective about W .

□

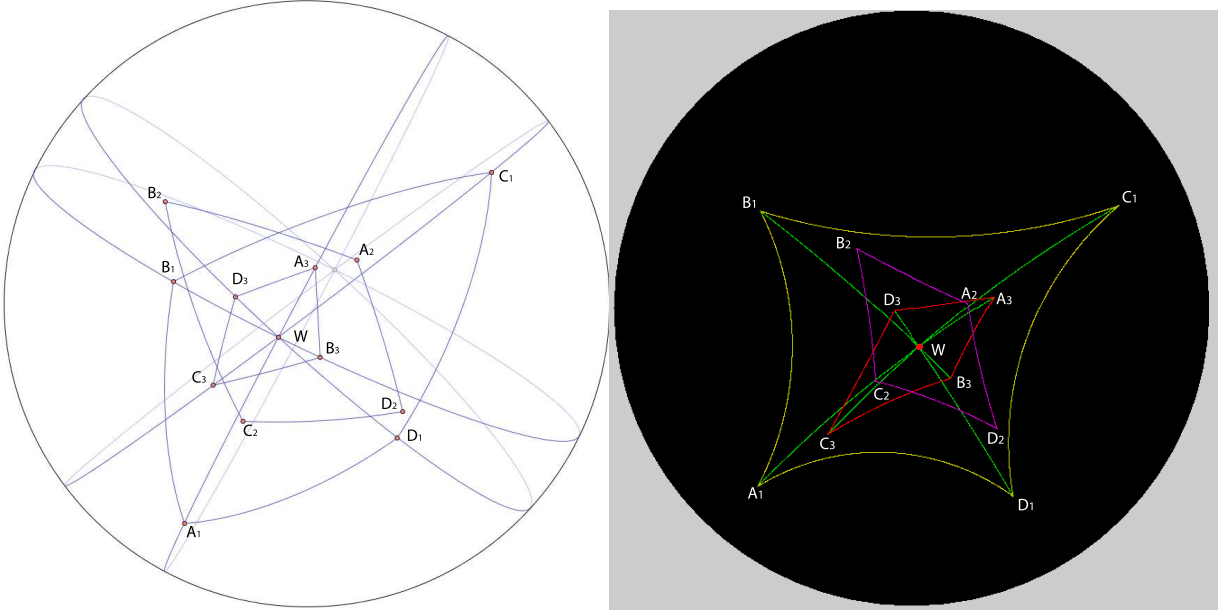


FIGURE 4.2. Two special cases of Theorem 4: On the left is \mathbb{S}^2 and on the right is 2-dimensional Hyperbolic space viewed in the Poincaré disk model. The points A_j, B_j, C_j and D_j are the members of the set $N^{(j)}$. The point W is the point about which $N^{(1)}$ and $N^{(3)}$ are in perspective.

From the proof, it is not hard to see that

Corollary 5. *In $(n-2)$ -dimensional Euclidean geometry, all sets of the form $N^{(2i+1)}$ are homothetic, and all sets of the form $N^{(2k)}$ are homothetic.*

We will now show how to reverse the iterative construction, so that given $N^{(i+1)}$ we can determine $N^{(i)}$. Recall that the isogonal conjugate of a point P with respect to a triangle $\triangle ABC$ in the plane is the point of intersection of the three lines obtained by reflecting line PA in the angle bisector of $\angle A$, line PB in the angle bisector of $\angle B$ and line PC in the angle bisector of $\angle C$. In case that P lies on the circumcircle of $\triangle ABC$, the isogonal conjugate is an ideal point. For a more thorough discussion of isogonal conjugation in \mathbb{R}^2 and \mathbb{R}^3 , we refer the reader to [12] and [13] respectively.

For our purposes, we will not be using this definition of the isogonal conjugate due to the ease and generality of the following definition, which is equivalent to the former in \mathbb{R}^2 and \mathbb{R}^3 :

Definition. Let $S = P_1 \cdots P_{d+1}$ be a simplex in d -dimensional space V , P be a point not equal to P_1, \dots, P_{d+1} and P'_1, \dots, P'_{d+1} be the $d+1$ reflections of P in the facets of S . Then the isogonal conjugate of P with respect to S , denoted by $Isos_S P = Iso_{P_1 \cdots P_{d+1}} P$, is the center of the hypersphere $(P'_1 \cdots P'_{d+1})$.

The following property shows that isogonal conjugation is an involution.

Lemma 6. *With respect to any simplex S in Euclidean, Hyperbolic or Elliptic geometry, $Iso_S Iso_S P = P$.*

Proof. Let $Q = Iso_S P$ and for $i = 1, \dots, d+1$, let P'_i and Q'_i be the reflection of P , respectively Q , in the facet opposite to P_i . We then have $PQ'_i = QP'_i$ for each $i = 1, \dots, d+1$. As Q is the center of the hypersphere $(P'_1 \cdots P'_{d+1})$, we also have $QP'_i = QP'_j$ for every $i, j \in \{1, \dots, d+1\}$. Since $PQ'_j = QP'_j$, $PQ'_j = QP'_i = PQ'_i$, so that P is equidistant from all the Q'_k . Therefore P is the center of the hypersphere $(Q'_1 Q'_2 \cdots Q'_{d+1})$. \square

Recall that we are using the notation that $V_i^{(2)} = V_{i+n}^{(2)}$. The following Theorem allows us to reverse the iterative process:

Theorem 7. *In n -dimensional Euclidean, Hyperbolic and Elliptic geometry, $Iso_{V_{i+1}^{(2)} \cdots V_{n+i-1}^{(2)}} V_i^{(2)} = V_i$.*

Proof. Consider the reflections of the vertex V_i in each of the facets of the simplex $V_{i+1}^{(2)} \cdots V_{n+i-1}^{(2)}$. Since the facets of this simplex are the mediators $PB(V_i V_j)$, $\forall j \in \{1, \dots, n\} \setminus \{i\}$, reflecting V_i in them results in the points V_j , $\forall j \in \{1, \dots, n\} \setminus \{i\}$. The center of the hypersphere $(V_{i+1} \cdots V_{n+i-1})$ is by definition $V_i^{(2)}$, so that $Iso_{V_{i+1}^{(2)} \cdots V_{n+i-1}^{(2)}} V_i = V_i^{(2)}$. By lemma 6, $Iso_{V_{i+1}^{(2)} \cdots V_{n+i-1}^{(2)}} V_i^{(2)} = V_i$. \square

We will now shift our attention from the sets $N^{(i)}$ to the polytopes $P^{(i)}$ with vertices $N^{(i)}$.

Definition. Two polytopes P and P' are said to be *combinatorially equivalent* (or of the same *combinatorial type*) provided there exists a bijection ϕ between the set $\{F\}$ of all faces of P and the set $\{F'\}$ of all faces of P' , such that $F_1 \subset F_2$ if and only if $\phi(F_1) \subset \phi(F_2)$ [14].

We will say that $P^{(i)} \sim P^{(j)}$ when a polytope $P^{(i)}$ with vertices $N^{(i)}$ can be chosen to have the same combinatorial type as a polytope $P^{(j)}$ with vertices $N^{(j)}$, and $P^{(i)}$ and $P^{(j)}$ are similar.

Corollary 5 then implies that in $(n-2)$ -dimensional Euclidean geometry,

$$P^{(1)} \sim P^{(3)} \sim P^{(5)} \sim \dots$$

and

$$P^{(2)} \sim P^{(4)} \sim P^{(6)} \sim \dots$$

Let $|P^{(i)}|$ denote the volume of $P^{(i)}$. From Corollary 5 it follows that for all i, j and k ,

$$\frac{|P^{(i)}|}{|P^{(i+2k)}|} = \frac{|P^{(j)}|}{|P^{(j+2k)}|}.$$

In the case $d = 2$, it is also true that $\frac{|P^{(i)}|}{|P^{(i+k)}|} = \frac{|P^{(j)}|}{|P^{(j+k)}|}$. In fact, it is shown in [8] that

$$\frac{|P^{(k+1)}|}{|P^{(k)}|} = \frac{1}{4}(\cot \alpha + \cot \gamma) \cdot (\cot \beta + \cot \delta),$$

where α, β, γ and δ are the angles of the quadrilateral $P^{(1)}$. However, experiment shows that the ratio of volumes of consecutive polyhedra is not in general only dependent on $P^{(1)}$. Another property that holds for $d = 2$, but not generally, is that if $P^{(1)}$ is nondegenerate and noncyclic, then $P^{(2)}$ is never cyclic. An easy way to see this is by applying isogonal conjugation as in lemma 7, which shows that $P^{(1)}$ must be at infinity. On the other hand, for $d = 3$, we can construct an example where $P^{(1)}$ is nondegenerate and nonconhyperspherical and $P^{(2)}$ is conhyperspherical by using the same lemma 7 (see figure 4.3) because the isogonal conjugate of a point on the circumsphere is not in general at infinity.

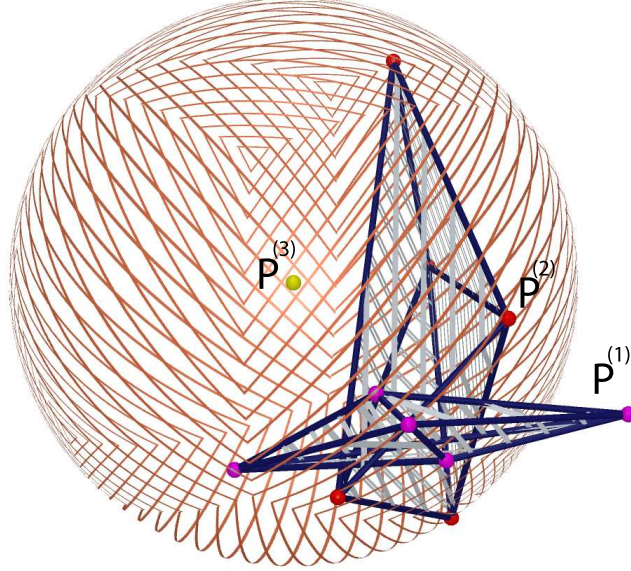


FIGURE 4.3. The polyhedron $P^{(1)}$ is constructed to be not conhyperspherical. The polyhedron $P^{(2)}$ obtained from $P^{(1)}$ via the Perpendicular Bisector Construction, on the other hand, is inscribed in a sphere. The next generation polyhedron $P^{(3)}$ is the center of the sphere. This phenomenon that $P^{(i)}$ is noncyclic but $P^{(i+1)}$ is cyclic cannot occur in \mathbb{R}^2 .

5. THE ISOPTIC POINT IN \mathbb{R}^d

We now show that any pair of odd and any pair of even generation polytopes are homothetic about the same point:

Theorem 8. *The center of homothety $W^{(1)}$ of any pair of polytopes $P^{(2i+1)}, P^{(2j+1)}$ coincides with the center of homothety $W^{(2)}$ of any pair of polytopes $P^{(2k)}, P^{(2l)}$.*

Proof. Let M be the midpoint of segment $V_a V_b$ and $M^{(3)}$ that of $V_a^{(3)} V_b^{(3)}$. For $c \notin \{a, b\}$, $V_c^{(2)}$ lies on the perpendicular bisector of $V_a V_b$, so that $V_a M V_c^{(2)}$ forms a right triangle. Similarly, $V_a^{(3)} M^{(3)} V_c^{(4)}$ is a right triangle. Since $P^{(1)} \sim P^{(3)}$, and V_c ($V_c^{(3)}$) is the center of the hypersphere through all V_i ($V_i^{(3)}$), $i \in \{1, \dots, n\} \setminus \{c\}$, the two triangles are similar, hence homothetic. Therefore $W^{(1)} = V_b V_b^{(3)} \cap M^{(1)} M^{(3)} \cap V_c^{(2)} V_c^{(4)}$. Now consider $V_d^{(2)}$ and $V_d^{(4)}$ in place of $V_c^{(2)}$ and $V_c^{(4)}$ for some $d \notin \{a, b, c\}$. Then by the same reasoning, $W^{(1)} = V_b V_b^{(3)} \cap M^{(1)} M^{(3)} \cap V_d^{(2)} V_d^{(4)}$. But $W^{(2)} = V_c^{(2)} V_c^{(4)} \cap V_d^{(2)} V_d^{(4)}$. Therefore $W^{(1)} = W^{(2)}$. \square

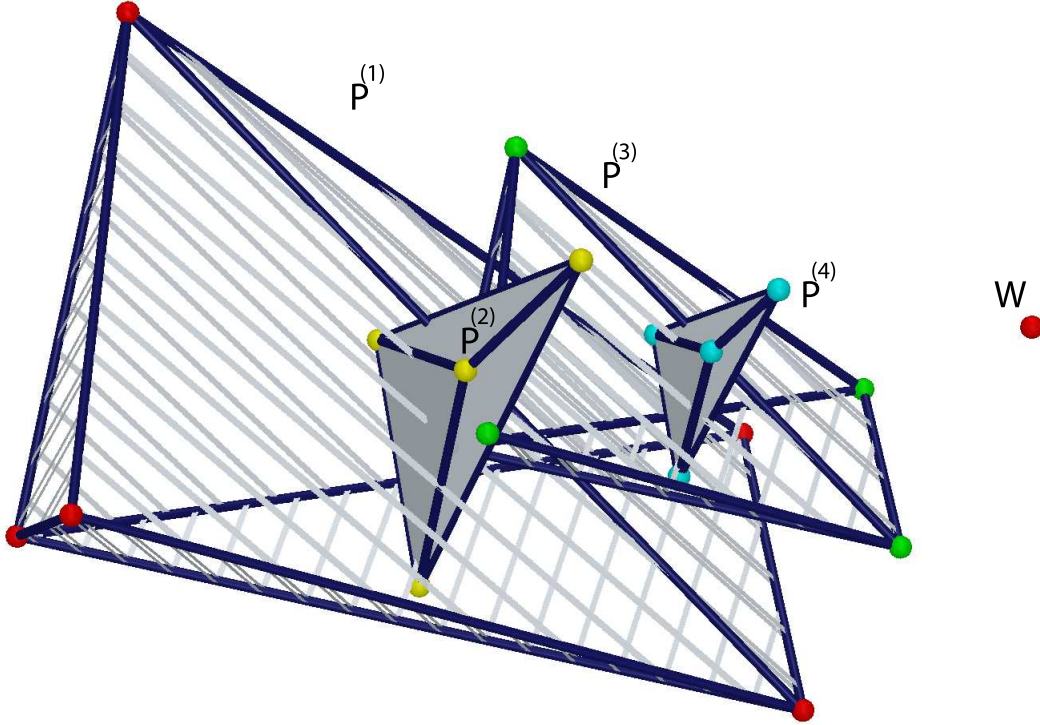


FIGURE 5.1. An example illustrating Theorem 8: The polyhedra $P^{(1)}$ and $P^{(3)}$ and the polyhedra $P^{(2)}$ and $P^{(4)}$ in \mathbb{R}^3 are homothetic about the same point W .

We will call this “universal” center of homothety W . This point can be seen as the limit of the construction when $\frac{|P^{(1)}|}{|P^{(3)}|} > 1$ and the limit of the reverse construction when $\frac{|P^{(1)}|}{|P^{(3)}|} < 1$.

In the case $d = 2$, this point is called the *Isoptic point* due to its property of subtending equal angles at each triad circle of the quadrilateral (see Radko and Tsukerman [8]). In \mathbb{R}^2 , W has many properties that are analogous to those of the circumcenter. More generally, if $N^{(1)}$ is approaching a conhyperspherical configuration, then the limit of W is the circumcenter of $N^{(1)}$.

Finally, we pose the following problem. It is shown in [8] that in \mathbb{R}^2 , the ratio of similarity of $P^{(i)}$ to $P^{(i+2)}$ is equal to the following expressions:

$$\begin{aligned} \frac{1}{4}(\cot \alpha + \cot \gamma) \cdot (\cot \beta + \cot \delta) &= \frac{1}{4}(\cot \alpha_1 - \cot \beta_2) \cdot (\cot \delta_2 - \cot \gamma_1) \\ &= \frac{1}{4}(\cot \delta_1 - \cot \alpha_2) \cdot (\cot \beta_1 - \cot \gamma_2), \end{aligned}$$

where the angles $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1, 2$, are the angles formed between sides and diagonals of a quadrilateral (see figure 5.2) and $\alpha = \alpha_1 + \alpha_2$, $\beta = \beta_1 + \beta_2$, etc. Is there a similar expression for the ratio of similarity in \mathbb{R}^n ?

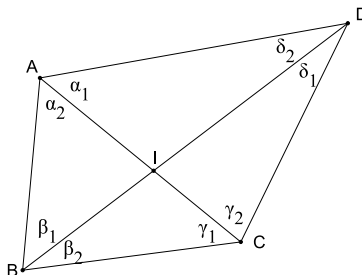


FIGURE 5.2. The angles between the sides and diagonals of a quadrilateral.

REFERENCES

- [1] J. Langr, problem 1085E, *Am. Math. Monthly*, **60** (1953), 551.
- [2] C.S. Ogilvy, *Tomorrow's Math. Unsolved Problems for the Amateur*, Oxford University Press, New York, 1962. (2nd edn., 1972. Pages reference second edition).
- [3] A. Bogomolny, *Quadrilaterals Formed by Perpendicular Bisectors*, Interactive Mathematics Miscellany and Puzzles, <http://www.cut-the-knot.org/Curriculum/Geometry/PerpBisectQuadri.shtml>, Accessed 06 December 2010.
- [4] B. Grünbaum, *Quadrangles, Pentagons, and Computers*, *Geombinatorics* 3 (1993), 4-9.
- [5] D. Bennett, *Dynamic Geometry Renews Interest in an Old Problem*, *Geometry Turned On*, MAA Notes 41, 1997, 25-28.
- [6] J. King, *Quadrilaterals Formed by Perpendicular Bisectors*, in *Geometry Turned On*, MAA Notes 41, 1997, 29-32.

- [7] G.C. Shephard, *The perpendicular bisector construction*, Geom. Dedicata, **56**, 75-84 (1995).
- [8] O. Radko and E. Tsukerman, "*The Perpendicular Bisectors Construction, the Isoptic Point and the Simson Line of a Quadrilateral*", to appear in *Forum Geometricorum*.
- [9] Weisstein, Eric W. "Polar." From MathWorld—A Wolfram Web Resource.
<http://mathworld.wolfram.com/Polar.html>
- [10] J. N. Cederberg, A Course in Modern Geometries, 2nd edition, p. 283-298.
- [11] P. O. Bell, Generalized Theorems of Desargues for n -Dimensional Projective Space, Proceedings of the American Mathematical Society Vol. 6, No. 5 (Oct., 1955), pp. 675-681.
- [12] D. Grinberg, Isogonal conjugation with respect to triangle, unpublished notes,
<http://www.cip.i.lmu.de/~grinberg/Isogonal.zip>, last accessed on 1/24/2010.
- [13] N. A. Court, Modern Pure Solid Geometry. The Macmillan Co., 1935.
- [14] B. Grünbaum, Convex Polytopes, 2nd edition, p. 38.

Emmanuel Tsukerman: Stanford University

E-mail address: emantsuk@stanford.edu